Math 259A Lecture 20 Notes

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1 Distinguishing Group von Neumann Algebras

1.1 $L(\mathbb{F}_2)$ and $L(S_\infty)$ are nonisomorphic

We showed that if Γ is an ICC group, then $L(\Gamma)$ is a II_1 factor. We have many examples of ICC groups.

Example 1.1. S_∞, the group of finite permutations of $\mathbb{N}$ is ICC.

Example 1.2. $\mathbb{F}_n$, the free group on $n \geq 2$ elements, is ICC.

Example 1.3. If $H \neq 1$ and Γ_0 is an infinite group, the wreath product of H and Γ_0 is ICC.

It is not clear that different groups gives different II_1 factors. After all, there is only 1 kind of type I_∞ factor, $\mathcal{B}(\ell^2(\mathbb{N}))$.

Recall that if a factor M has a trace state, then M is a finite factor. Later, we will show that this is an iff.

Definition 1.1. A II_1 factor M (with a trace state τ) has **property Gamma** if for all $x_1, \dots, x_n \in M$ and $\varepsilon > 0$, there exists some $u \in U(M)$ such that $\tau(u) = 0$ and $\|ux_iu^* - x_i\|_\tau < \varepsilon$ for all i.

Here, the norm is $\|x\|_\tau = \tau(x^*x)^{1/2}$. This comes from an inner product, so we may call this $\|x\|_2$.

Proposition 1.1. *If Γ Is locally finite and ICC, then $L(\Gamma)$ has property Gamma.*

Proof. $L(\Gamma) = \{\sum c_g u_g : c_g \in \mathbb{C}, \ell^2 \text{ summable convolvers}\}$. Then $\mathbb{C}\Gamma$ is a *-subalgebra. If $x_1^0, \dots, x_n^0 \in \mathbb{C}\Gamma$, then take a finite subgroup containing them. Now we can pick a unitary convolver outside of this finite subgroup. $\square$

Proposition 1.2. $L(\mathbb{F}_2)$ *does not have property Gamma.*

To prove this, we will prove a lemma.

Lemma 1.1. *Let Γ be an ICC group. Assume exists a set $S \subseteq \Gamma$ and $g_1, g_2, g_3 \in \Gamma$ such that*

1. $S \cup g_1 S g_1^{-1} \cup \{e\} = \Gamma$,
2. S, $g_2 S g_2^{-1}, g_3 S g_3^{-1}$ *are disjoint.*

Then $L(\Gamma)$ does not have property Γ.

Proof. Assume $L(\Gamma)$ has property Gamma. So for any $\varepsilon > 0$, there is a $u \in U(L(\Gamma))$ with $\tau(u) = 0$, $u = \sum c_g u_g$, $c_e = 0$, and $\|u u_{g_i} u^* - u_{g_i}\|_2 < \varepsilon$ for $i = 1, 2, 3$. This says that $\sum_{g \in \Gamma} |c_{g_i g g_i^{-1}} - c_g|^2 < \varepsilon^2$ for $i = 1, 2, 3$. For any $F \subseteq \Gamma$, denote $\nu(F) = \sum_{g \in G} |c_g|^2$ (so $\nu(\Gamma) = 1$). Then, by the triangle inequality in $\|\cdot\|_{\ell^2(S)}$,

$$\left| \left(\sum_{g \in S} |c_g|^2 \right)^{1/2} - \left(\sum_{g \in S} |c_{g_i g g_i^{-1}}|^2 \right)^{1/2} \right| \leq \left(\sum_{g \in S} |c_g - c_{g_i g g_i^{-1}}|^2 \right)^{1/2} < \varepsilon.$$

That is, $|\nu(S)^{1/2} - \nu(g_i S g_i^{-1})^{1/2}| < \varepsilon$. So

$$|\nu(S) - \nu(g_i S g_i^{-1})| \leq 2\varepsilon.$$

But by property (1),

$$\begin{aligned} \nu(\Gamma) &\leq \nu(S) + \nu(g_1 S g_1^{-1}) + \cancel{\nu(\{e\})} \\ &\leq \nu(S) + \nu(S) + 2\varepsilon \\ &= 2\nu(S) + 2\varepsilon \end{aligned}$$

By property (2), we have

$$1 \geq \nu(S) + \nu(g_2 S g_2^{-1}) + \nu(g_3 S g_3^{-1}) \geq 3\nu(S) - 4\varepsilon.$$

This is a contradiction. $\square$

Now we can prove the proposition.

Proof. Let $\Gamma = \mathbb{F}_2$ with S, the set of words that start with a^n for $n \neq 0$. Then take $g_1 = a$, $g_2 = b$, and $g_3 = b^{-1}$. These satisfy properties (1) and (2), so by the lemma, $\mathbb{F}_2$ does not have property Gamma. $\square$

Remark 1.1. This kind of partition of a group is generally called a **paradoxical partition**. This is a similar kind of thing as what happens in the Banach-Tarski paradox. In that case, $\mathrm{SO}(3) \supseteq \mathbb{F}_2$, and we use this paradoxical partition in that proof.

Corollary 1.1. *$L(\mathbb{F}_2) \neq L(S_\infty)$.*

1.2 Loss of information from forming $L(\Gamma)$ from Γ

However, this proof is very ad-hoc. It is difficult to tell apart the structure of $L(\Gamma)$ for different groups Γ. The functor $\Gamma \mapsto \mathbb{C}\Gamma$ loses some information. But then $\Gamma \mapsto \overline{\mathbb{C}\Gamma} = L(\Gamma)$ loses a lot of information!

Proposition 1.3. *$\mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2]$ and $\mathbb{C}\mathbb{Z}_4$ are both isomorphic to $\mathbb{C}^4$.*

This is because of the torsion. In fact, we have the following fact:

Proposition 1.4. *Let Γ be abelian and countably infinite. Then there is a *-algebra isomorphism $(L(\Gamma), \tau) \cong (L^\infty([0,1]), \int \cdot \, dm)$*

This loss of information happens when going from $\mathbb{C}\Gamma \mapsto L(\Gamma)$.

Proposition 1.5. *$\mathbb{C}\mathbb{Z}^n$ are nonisomorphic for different n.*

Proof. The invertible elements in $\mathbb{C}\mathbb{Z}^n$ are $\mathbb{Z}^n(\mathbb{C} \setminus \{0\})$. □

Here is a conjecture:

Theorem 1.1 (Kaplansky)**.** *If Γ is torsion free, then* $\mathrm{Inv} = \Gamma \cdot (\mathbb{C} \setminus \{0\})$.

This is true if Γ is an **orderable** group. In fact, $\mathbb{F}_n$ is orderable, and many amenable groups are orderable.

Definition 1.2. If Γ is a group, its **group C^*-algebra** is $C_r^*(\Gamma) := \overline{\mathbb{C}(\Gamma)}^{\text{norm}} = \operatorname{span} \overline{\lambda(\Gamma)}^{\text{norm}}$.

$C^*(\Gamma)$ has lots and lots of unitary elements.

Proposition 1.6. *Suppose Γ is abelian and torsion-free. If U_0 is the connected component of 1, $U(C_r^*)/U_0 \cong \Gamma$.*

So this algebra does remember the group.

1.3 Amenable groups

The real property we care about here is amenability. Here is a definition due to von Neumann in the 30s:

Definition 1.3. A group Γ is **amenable** if it has an **invariant mean**, i.e. a state φ on $\ell^\infty(\Gamma)$ such that $\varphi(g^{-1}f) = \varphi(f)$ for all $f \in \ell^\infty$ and $g \in \Gamma$ ($\Gamma \circlearrowright \ell^\infty(\Gamma)$ by left translation on coordinates).

Example 1.4. $\mathbb{Z}^n$ is amenable for any n.

Example 1.5. S_∞ is amenable.

Definition 1.4. Γ has **Følner's property** if for any nonempty, finite $F \subseteq \Gamma$ and $\varepsilon > 0$, there exists a finite $K \subseteq \Gamma$ such that

$$\frac{|FK \triangle K|}{|K|} < \varepsilon.$$

This is same as saying that

$$\frac{|gK \triangle K|}{|K|} < \varepsilon \qquad \forall g \in F.$$

Theorem 1.2. *The Følner property implies ammenability.*

Proof. If Γ has F and is countable, then there exists a sequence $K_n \subseteq \Gamma$ with

$$\frac{|g_i K_n \triangle K_n|}{|K_n|} \xrightarrow{n\to\infty} 0.$$

Choose a non-principal ultrafilter ω on $\mathbb{N}$, and define $\varphi(f) = \lim_{n\to\omega} \frac{1}{|K_n|} \sum_{g \in K_n} f(g)$. This is called a **Banach limit**. So $f \mapsto \varphi(f)$ is linear from $\ell^\infty(\Gamma) \to \mathbb{C}$, $\varphi(1) = 1$, and $\varphi(g_i^{-1} f) = \varphi(f)$ for all i. $\square$

Remark 1.2. We only need to show that $\varphi(g_i^{-1} f) = \varphi(f)$ for the generators of the group.

Example 1.6. $\mathbb{Z}$ is amenable because the sets $K_n = [-n, n]$ gives it the Følner property.

Example 1.7. Locally finite groups are amenable because they satisfy the Følner property.

Proposition 1.7. *If a collection of groups H_i is amenable, then $\bigoplus_i H_i$ is amenable.*

Example 1.8. $\mathbb{Z} \rtimes \mathbb{Z}^n$ and $\mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}^n$ are ICC and amenable.

Theorem 1.3 (Murray-von Neumann, 1943)**.** *All locally finite ICC groups give the same II_1 factor. In fact, all AFD factors are isomorphic to $L(S_\infty)$.*

Definition 1.5. A II_1 factor M with a trace τ is called **approximately finite dimensional (AFD)** if given any $x_1, \ldots, x_n$ and $\varepsilon > 0$, there exists a finite dimensional von Neumann algebra $B \subseteq M$ and $y_1, \ldots, y_n \in B$ such that $\|x_i - y_i\| < \varepsilon$ for all i

Proposition 1.8. *If Γ is locally finite, then $L(\Gamma)$ is AFD.*

We also have the following remarkable theorem:

Definition 1.6. A II_1 factor is **amenable** if it has an invariant mean (or a **hypertrace**).

Theorem 1.4 (Connes, 1976)**.** *All II_1 factors M that are amenable are isomorphic to $L(S_\infty)$.*

Proposition 1.9. *$L(\Gamma)$ is amenable if and only if Γ is amenable.*

Proposition 1.10. *$\mathbb{F}_2$ is not amenable.*

This gives another proof that S_∞ and $\mathbb{F}_2$ have different group von Neumann algebras.

Corollary 1.2. *$L(\mathbb{F}_2) \ncong L(S_\infty)$.*